

## বিদ্যাসাগর বিশ্ববিদ্যালয় VIDYASAGAR UNIVERSITY

## Question Paper

## B.Sc. Honours Examinations 2020

(Under CBCS Pattern)
Semester - V
Subject: MATHEMATICS
Paper: C11T
(Partial Differential Equations \& Applications)
Full Marks : 60
Time : 3 Hours

Candiates are required to give their answer in their own words as far as practicable. The figures in the margin indicate full marks.

Attempt any three questions.
$3 \times 20=60$

1. (a) Find the integral surface passing through the curve $y^{2}+z^{2}=1, x+z=2$ and corresponding to the PDE $4 y z p+q=-2 y$. 10
(b) (i) Find PDE corresponding to the equation $z=x y+f\left(x^{2}+y^{2}\right), f$ being an arbitrary function.
(ii) Find the PDE of the family of right circular cone whose axis coincides with z axis.
2. (a) Reduce the PDE $\frac{\partial^{2} z}{\partial t^{2}}=c^{2} \frac{\partial^{2} z}{\partial x^{2}}$ to $\frac{\partial^{2} z}{\partial u \partial v}=0$ by $u=x-c t, v=x+c t$.
(b) (i) Solve the PDE by Lagrange's method $p y+q x=x y z^{2}\left(x^{2}-y^{2}\right)$.
(ii) Solve the PDE $p x+q y=z(1+p q)^{1 / 2}$.
3. (a) Solve the following one dimensional heat equation

$$
\frac{\partial T}{\partial t}-k \frac{\partial^{2} T}{\partial x^{2}}=0,0 \leq x \leq l, t \geq 0
$$

Subject to the condition
(i) $T(x, 0)=f(x)=l-x, 0 \leq x \leq l$
(ii) $T(0, t)=T(l, t)=0, t \geq 0$
(iii) $T(x, t)<\infty$ as $t \rightarrow \infty$.

Hence evaluate $\lim _{t \rightarrow \infty} T(x, t)$.
where $k$ is a constant.
(b) Find the general solution of the PDE $x\left(y^{n}-z^{n}\right) p+y\left(z^{n}-x^{n}\right) q=z\left(x^{n}-y^{n}\right) . \quad 5$
4. (a) Find the solution of the following two-dimensional Laplace Equation at any interior of the rectangle $0 \leq x \leq a, 0 \leq y \leq b, \frac{\partial^{2} \varphi}{\partial x^{2}}+\frac{\partial^{2} \varphi}{\partial y^{2}}=0$, subject to the boundary conditions $\varphi_{x}(0, y)=\varphi_{y}(a, y)=0,0 \leq y \leq b$ and $\varphi_{y}(x, 0)=0 ; \varphi_{y}(x, b)=f(x), 0 \leq x \leq a$.
(b) Find the complete integral of the PDE $z^{2}=\frac{\partial z}{\partial x} \frac{\partial z}{\partial y} x y$, by Charpit's method.
5. (a) Solve the following equation for a string of finite length $u_{t t}-9 u_{x x}=0,0 \leq x \leq 2, t \geq 0$. Subject to the boundary conditions $u(0, t)=u_{t}(0, t)=0, u(2, t)=u_{t}(2, t)=0, t \geq 0$ and the initial condition $u(x, 0)=x, u_{t}(x, 0)=0,0 \leq x \leq 2$.
(b) The general solution of the equation $\left(D^{2}-2 D D^{\prime}+D^{\prime 2}\right) u=e^{x+2 y}$.
6. (a) Solve the one dimensional wave equation of infinite string $u_{t t}-c^{2} u_{x x}=0,0 \leq x \leq \infty, t \geq 0$
subuject to the initial coditions $u(x, 0)=f(x), u_{t}(x, 0)=g(x), x \geq 0$ and the boundary condition $u(0, t)=0, t \geq 0$.
(b) Find the P.I of the equation $\left(D-D^{\prime}\right)^{2} z=\tan (x+y)$.

leaving the vertex is given by the equation $\tan \theta=\tanh (\sqrt{\mu} t), \mu$ is the acc. at distance unity.
(b) Obtain the solution of the wave equation $u_{t t}=c^{2} u_{x x}$ under the following conditions:
(i) $u(0, t)=u(2, t)=0$
(ii) $\quad u(x, 0)=\sin ^{3} \frac{\pi x}{2}$
(iii) $u_{t}(x, 0)=0$
3. (a) Prove that the solution of one-dimensional diffusion equation in the region $0 \leq x \leq \pi$, subject to the condition :
(i) $u(x, t)$ is finite as $t \rightarrow \infty$
(ii) $\quad u(0, t)=0=u(\pi, t)$
(iii)

$$
u(x, 0)=\left\{\begin{align*}
x, & 0 \leq x \leq \frac{\pi}{2}  \tag{10}\\
\pi-x, & \frac{\pi}{2} \leq x \leq \pi
\end{align*} \text { is } u(x, t)=\frac{4}{\pi} \sum_{n=1}^{\infty} \frac{e^{a n^{2} t} \sin \left(\frac{n \pi}{2}\right)}{n^{2}} \sin (n x) .\right.
$$

(b) Define Dirac delta function with a brief explanation.
4. (a) Find the canonical form of the PDE $y^{2} u_{x x}+2 y u_{x y}+u_{y y}-u_{y}=0$.
(b) A bead slides down a rough circular wire, which is in a vertical plane, starting from rest at the end of the horizontal diameter. When it has described an angle $\theta$ about the centre, show that the square of the angular velocity is $\frac{2 g}{a\left(1+4 \mu^{2}\right)}\left\{\left(1-2 \mu^{2}\right) \sin \theta+3 \mu\left(\cos \theta-e^{-2 \mu \theta}\right)\right\}$ where $\mu$ is the coeficient of friction and $a$ is the radius of the circle.
5. (a) A particle of mass $M$ is at rest and begins to move under the action of constant force F in a fixed direction. It encounters the resistance of a stream of fine dust moving in the opposite direction with velocity V , which deposits matter on it at a constant rare $\rho$. Show that its mass will be $m$ when it has travelled a distance

$$
\begin{equation*}
\frac{k}{\rho^{2}}\left[m-M\left\{1+\log \frac{m}{M}\right\}\right], \text { where } \mathrm{K}=\mathrm{F}-\rho \mathrm{V} \tag{7}
\end{equation*}
$$

(b) Find the solution of Laplace's equation $\nabla^{2} \varphi=0$ in the semifinite region bounded by $x \geq 0,0 \leq y \leq 1$ subject to the boundary conditions $\left(\frac{\partial \varphi}{\partial x}\right)_{x=0}=0,\left(\frac{\partial \varphi}{\partial y}\right)_{y=0}=0$ and $\varphi(x, 1)=f(x)$ where $f(x)$ is assumed to be known.
6. (a) Solve : $p x+q y=z \sqrt{1+p q}$ by charpit's method.
(b) A particle describes an ellipse under a force $\frac{u}{(\text { distance })^{2}}$ towards the focus; if it was projected with velocity V from a point distance r from the centre of the force, show that its periodic time is $\frac{2 \pi}{\mu} \cdot\left[\frac{2}{r}-\frac{V^{2}}{\mu}\right]^{-\frac{3}{2}}$.
(c) From a partial diferential equation by eliminating the arbitrary function $\varphi$ from

$$
\begin{equation*}
\varphi\left(x^{2}+y^{2}+z^{2}, z^{2}-2 x y\right)=0 . \tag{3}
\end{equation*}
$$

7. (a) Discuss the solution of the two dimensional wave equation $\frac{\partial^{2} u}{\partial x^{2}}+\frac{\partial^{2} u}{\partial y^{2}}=\frac{1}{c^{2}} \frac{\partial^{2} u}{\partial t^{2}}$ by the method of separation of variables.
(b) A comet is moving in a parabola about the Sun as focus; when at the end of its latus rectum its velocity suddenly becomes altered in the ratio $n: 1$, where $n<1$; show the comet will describe an ellipse whose eccentricity is $\sqrt{1-2 n^{2}+2 n^{4}}$ and whose major axis $\frac{1}{1-n^{2}}$, where $2 l$ is the latus rectus of the parabolic path. 7
8. (a) A partiocle is acted on by a central repulsive force, which varies as the nth power of the distance; if the velocity at any point of the path be equal to that, which would be acquired in falling from the centre to the point, show that the equation to the path is of the form $r^{\frac{n+3}{2}} \cos \frac{n+3}{2} \theta=$ constant.
(b) Find the equation of the integral surface of the diferential equation $2 y(z-3) p+(2 x-z) q=y(2 x-3)$ which passes through the circle $z=0, x^{2}+y^{2}=2 x$.
(c) Find the general solution of PDE $x^{2} \frac{\partial z}{\partial x}+y^{2} \frac{\partial z}{\partial y}=(x+y) z$.

## Group-B

Answer any six questions :
9. (i) From a PDE when $\varphi(u, v)=0$, where $u=x+y+z, v=x^{2}+y^{2}+z^{2}$.
(ii) What is the interpretation that $z=f(x, y)$ is the integral surface of $p P+q Q=R$.
(iii) Find the characteristics of the equation $u_{x x}+2 u_{x y}+\sin ^{2} x u_{y y}+u_{y}=0$.

When it is hyperbolic.
(iv) Define 'Dirichlet boundary condition' and 'Neumann boundary condition'.
(v) If a particle moves on a curve $\sqrt{r} \cos \frac{\theta}{2}=\sqrt{a}$ with cross-radial velocity constant then show that the velocity of the particle is constant.
(vi) Give the geometrical interpretation of Cauchy IVP $u_{t}+c u_{x}=0, x \in R, t>0$ where $u(x, 0)=f(x), x \in R$.
(vii) Write the diferent types of first order PDE with standar form.
(viii) Verify the equation $z=\sqrt{2 x+a}+\sqrt{2 y+b}$ is a complete integral of the PDE $\frac{1}{z}=\frac{1}{p}+\frac{1}{q}$.
(ix) Show that if $f$ and $g$ are arbitrary function of their respective arguments then $u=f(x-v t+i \alpha y)+g(x-v t-i \alpha y)$ is a solution of $\frac{\partial^{2} u}{\partial x^{2}}+\frac{\partial^{2} u}{\partial y^{2}}=\frac{1}{c^{2}} \frac{\partial^{2} u}{\partial t^{2}}$ where $\alpha^{2}=1=\frac{v^{2}}{c^{2}}$.
(x) A particle describes a curve $s=c \tan \Psi$ with uniform speed $v$. Find the acceleration indicating its direction.

2. (a) Suppose $G$ is a cyclic group. Then prove that every subgroup of $G$ is a characteristic subgroup of $G$.
(b) Express the Klein's four group as an internal direct product of two of its proper subgroups.
(c) Let $G^{\prime}$ denote the commutator subgroup of a group $G$. Prove that $G^{\prime}$ is a normal subgroup of $G$ and $G / G^{\prime}$ is abelian.
(d) Prove that $\operatorname{Aut}\left(\mathbb{Z}_{8}\right)$ is isomorphic to the Klein's four group.
(e) Let $G$ be a finite group and $H$ be a subgroup of $G$ such that $|H|=p^{k}$, where $p$ is a prime and $k$ is a non-negative integer. Then show that
$[G: H] \equiv_{p}[N(H): H]$
where $N(H)=\left\{g \in G: g H g^{-1}=H\right\}$ is the normalizer of the subgroup $H$ in $G$. 6
3. (a) Prove that $S_{3}$ cannot be expressed as an internal direct product of its proper subgroups.
(b) Is the group $\mathbb{Z}_{2} \times \mathbb{Z}_{2} \times \mathbb{Z}_{2}$ isomorphic to $\mathbb{Z}_{2} \times \mathbb{Z}_{4}$ ? Justify your answer.
(c) Find the comutator subgroup of $S_{3}$. Let $H, K$ be subgroups of $G$ such that $H \subseteq K$. If $H$ is a characteristic subgroup of $K$ and $K$ is a normal subgroup of $G$ then show that $H$ is normal in $G$.
(d) Prove that for any group $G,|G / Z(G)| \neq 91$.
(e) Let $G$ be a group of order $p n$ where $p$ is a prime and $p>n$. If $H$ is a subgroup of order $p$ of $G$ then show that $H$ is a normal subgroup of $G$.
4. (a) Let $G$ be a group. Give an example of a group action of $G$ on $G$.
(b) Let $G$ be a group acting on a non-empty set $S$. Prove that the stabilizer of $a$, where $a \in S$, is a subgroup of $G$.
(c) Show that $I_{3}=\{1,2,3\}$ is an $S_{3}$-set (i.e., the group $S_{3}$ acts on $I_{3}$ ) where the action is defined by $(\sigma, a) \rightarrow \sigma . a=\sigma(a)$ for all $\sigma \in S_{3}$ and for all $a \in I_{3}$. Find all the distinct orbits of $S_{3}$. Also find the stabilizers of 1,2 and 3 .
(d) Let $G$ be a finite group and $a \in G$. Prove that $[G: C(a)]=1$ if and only if $a \in Z(G)$ where $C(a)$ denotes the centralizer of $a$. 3
(e) State Sylow's first Theorem.
5. (a) Let $H$ be a subgroup of order 11 and index 4 of a group $G$. Does $G$ have a non-trivial proper normal subgroup ? Justify your answer.
(b) State class equation for a finite group.
(c) For any finite $p$-group $G, p$ is prime, show that $Z(G) \neq\{e\}$. If $G$ is a noncommutative group of order $p^{3}$ where $p$ is a prime, show that $|Z(G)|=p$.
(d) State Fundamental Theorem for finite abelian groups. Describe all abelian groups of order 360 up to isomorphism.
(e) Let $G$ be a finite $p$-group where $p$ is a prime. Then using Cauchy's Theorem prove that $|G|=p^{n}$.
(f) Let $G$ be a finite group having only two conjugacy classes. Show that $|G|=2$. 2
6. (a) Consider the left action of the group $G L(2, \mathbb{R})$ on $G L(2, \mathbb{R})$ by conjugation. Find the stabilizer of $\left(\begin{array}{ll}1 & 0 \\ 0 & 4\end{array}\right)$.
(b) Show that every group of order 99 has a normal subgroup of order 9 .
(c) For any finite $p$-group $G, p$ is prime, show that $Z(G) \neq\{e\}$. If $G$ is a noncommutative group of order $p^{3}$ where $p$ is a prime, show that $|Z(G)|=p$.
(d) Let $G$ be a group of order $273=$ 3.7.13. Show that $G$ has a cyclic subgroup of order 91.
(e) Using Sylow's theorems, prove that no group of order 56 is simple.

(b) (i) Let $H$ and $K$ be two characteristic subgroups of a group $G$. Prove that $H \cap K$ and $H K$ are also characteristic subgroups of $G$.
(ii) Prove that the direct product $\mathbb{Z}_{\mathrm{m}} \times \mathbb{Z}_{\mathrm{n}}$ is isomorphic with $\mathbb{Z}_{\mathrm{mn}}$ if and only if $m$ and $n$ are co-prime.
(iii) Let $G$ be a group acting on a non-empty set $S$ and $G_{x}$ denote the stabilizer of $x \in G$. Then prove that $G_{b x}=b G_{x} b^{-1}$ for any $b \in G$.

$$
(2+2)+4+4=12
$$

(c) (i) For any group (G;), prove that $\operatorname{Inn}(G)$ is a normal subgroup of $(\operatorname{Aut}(G), o)$ where $\operatorname{Inn}(G)$ denotes the set of all inner automorphisms of $G$.
(ii) Let $\alpha \in S_{4}$ be a $k$-cycle. Then prove that $\beta \in S_{4}$ is conjugate with $\alpha$ if and only if $\beta$ is also a $k$-cycle.
(iii) Using (ii) deduce class equation of $S_{4}$.
(d) (i) Let a be non-zero rational number. Prove that $\phi \in \operatorname{Aut}(\mathbb{Q},+)$ where $\phi:(\mathbb{Q},+)$ $\rightarrow(\mathbb{Q},+)$ is defined by $\phi(q)=q a$ for all $q \in \mathbb{Q}$. Hence prove that the only characteristic subgroups of $(\mathbb{Q},+)$ are $\{0\}$ and $\mathbb{Q}$.
(ii) Prove that if $G$ is a finite $p$-group then $Z(G)$ must be non-trivial.
(iii) Let $G$ be a group of order $p q$ where $p, q$ are both primes with $p>q$. If $q$ does not divide $(\mathrm{p}-1)$ then prove that $G$ is cyclic.
(e) (i) For any group (G;) prove that $G / Z(G)$ is isomorphic with $\operatorname{Inn}(G)$ where $\operatorname{Inn}(G)$ denotes the set of all inner automorphisms of $G$.
(ii) Find the number of elements o order 5 in the direct product $\mathbb{Z} 15 \times \mathbb{Z} 5$.
(iii) Let $G$ be a group and $H$ be a subgroup of $G$. Let $S=\{a H \mid a \in G\}$. Prove that there exists a homomorphism $\psi: G \rightarrow A(S)$ such that $\operatorname{ker} \psi \subseteq \mathrm{H}$.

$$
4+4+4=12
$$

(f) (i) Let $G$ be a finite group and $H$ be a subgroup of $G$ of index $p$, where $p$ is the smallest prime dividing $|G|$. Show that $H$ is a normal subgroup of $G$.
(ii) Show that every non-cyclic group o order 21 contains exactly 14 elements of order 3.
$6+6=12$
(g) (i) Describe all the abelian groups of order 1200 up to isomorphism.
(ii) Using Sylow's theorem, show that every group of order 45 has a normal subgroup of order 9 .
(iii) Prove that any group of order $p^{2}$ is abelian where $p$ is a prime.
(iv) Can $4=1+1+2$ be a class equation for a group? Justify your answer.

$$
5+2+3+2=12
$$

(h) (i) Define commutator subgroup of a group $G$. Find the commutator subgroup of $S_{3}$.
(ii) Give example (with explanation) of a non-cyclic commutative group of order 28.
(iii) Using Sylow's theorem, show that no group of order 108 is a simple group. $(2+3+2+5=12$

## Group-B

2. Answer any six questions:
(a) Prove that the set of all automorphisms of a group ( $G$;) form a group with respect to the composition of mappings.
(b) Define characteristic subgroup with an example.
(c) Consider two subgroups $H_{1}=\left\{e,\left(\begin{array}{ll}1 & 2\end{array}\right)\right\}$ and $H_{2}=\left\{e,\left(\begin{array}{lll}1 & 2 & 3\end{array}\right),\left(\begin{array}{lll}1 & 3 & 2\end{array}\right)\right\}$ of $S_{3}$. Can $S_{3}$ be expressed as an internal direct product of the subgropups $H_{1}$ and $H_{2}$ ? Justify your answer with proper reasons.
(d) Consider the direct product $S_{3} \times S_{3}$. Does it contain an element of order 9? Give reasons in support of your answer.
(e) Using the Fundamental theorem of finite abelian groups, describe all abelian groups o order $2^{4}$ up to isomorphism.
(f) Consider the conjugation action of the group $\left(\mathbb{Z}_{3},+\right)$ on itself. With respect to this group action, find all distinct orbits.
(g) Give an example of an infinite $p$-group.
(h) State Sylow's third theorem.
(i) Define a simple group with an example.
(j) Let $G$ be a group of order 69. Prove that $Z(G)$ is isomorphic with $\mathbb{Z}_{69}$.

(c) A plastic manufacturer has 1200 boxes of transparent wrap in stock at one factory and another 1000 boxes at its second factory. The manufacturer has order for this product from three different retailers, in quantities of 1000,700 and 1500 boxes respectively. The unit shipping cost from the factories to retailer are as follows :

| Retailer $\rightarrow$ <br> Factory <br> $\downarrow$ | I | II | III |
| :---: | :---: | :---: | :---: |
| A | 14 | 13 | 11 |
| B | 13 | 13 | 12 |

Determine the minimum cost shipping schedule for satisfying all demands from current stock. Formulate it to LPP.
(b) Is the system of equation
$x_{1}+x_{2}+x_{3}=4$
$2 x_{1}+5 x_{2}-2 x_{3}=3$
$x_{1}+7 x_{2}-7 x_{3}=5$
consistent? Justify your answer.
2. (a) Define the term extreme point of a convex set. What is special feature of this point?
(b) What do you mean by degeneracy of a simplex method? When does it occur?
(c) Show that basic feasible solutions are the extreme point of the convex set of feasible solutions of a LPP.
(d) State and prove the condition of unbounded solution of a maximization LPP, when we are going to solve it by simplex method.
3. (a) What do you mean by degeneracy in transportation problem.
(b) Describe the procedure to convert an assignment problem to a maximization problem.
(c) Find the optimal solution of the following LPP by solving its dual:

Minimize $Z=4 x_{1}+3 x_{2}+6 x_{3}$
Subject to the constraints $x_{1}+x_{2} \geq 2 ; x_{2}+x_{3} \geq 5 ; x_{1}, x_{2}, x_{3} \geq 0$.
(d) Show that the following problem has no feasible solution

Minimize $Z=x_{1}-3 x_{2}$

Subject to the constraints $x_{1}-x_{2} \geq 3 ;-x_{1}-x_{2} \geq 2 ; x_{1}, x_{2} \geq 0$.
4. (a) Show that in a balanced transportation problem, if the no. of source is $m$ and destination is $n$, no. of basic variables will be $m+n-1$.
(b) Why do we study duality in LPP ?
(c) A company has three plants $X, Y, Z$ and 3 warehourses $A, B$ and $C$. The supplies are transported from the plants to warehouses which are located at varying distance from the plants. On account of the varying distance, the trnasportation costs from plants to warehouses vary from Rs. 12 to Rs. 24 per unit. The company wishes to minimize the transportation costs. The costs in Rs. from the plants to warehouses are as shown below :

|  | A | B | C | Supply |
| :--- | :--- | :--- | :--- | :--- |
| $X$ | 12 | 8 | 18 | 400 |
| $Y$ | 20 | 10 | 16 | 350 |
| $Z$ | 24 | 14 | 12 | 150 |
| Demand | 500 | 200 | 3000 |  |

Determine the optimal shipping schedule.
(d) Solve the travelling salesman problem where the entries are given as distance. Find minimum distance.

|  | A | B | C | D | E |
| :---: | :---: | :---: | :---: | :---: | :---: |
| A | $\infty$ | 2 | 5 | 7 | 1 |
| B | 6 | $\infty$ | 3 | 8 | 2 |
| C | 8 | 7 | $\infty$ | 4 | 7 |
| D | 12 | 4 | 6 | $\infty$ | 5 |
| E | 1 | 3 | 2 | 8 | $¥$ |

5. (a) Find the dual of the following LPP:

Maximize $\quad Z=x_{1}-x_{2}+3 x_{3}+2 x_{4}$

Subject to $x_{1}+x_{2} \geq-1$

$$
\begin{aligned}
& x_{1}-3 x_{2}-x_{3} \leq 7 \\
& x_{1}+x_{3}-3 x_{4}=-2
\end{aligned}
$$

$x_{1}, x_{4} \geq 0$ and $x_{2}, x_{3}$ are unrestricted in sign.
(b) Why an assignment problem is not a LPP ?
(c) Find the optimal solution of the transportation problem using VAM method.

|  | $D_{1}$ | $D_{2}$ | $D_{3}$ | $D_{4}$ | $D_{5}$ |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $O_{1}$ | 10 | 6 | 4 | 3 | 3 | 41 |
| $O_{2}$ | 4 | 3 | 0 | 1 | 7 | 15 |
| $O_{3}$ | -1 | 4 | -3 | 0 | 2 | 23 |
|  | 19 | 10 | 4 | 8 | 15 |  |

(d) Find the optimal assignments for the assignment problem with the following cost matrix.

|  | $P$ | $Q$ | $R$ | $S$ | $T$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $A$ | 85 | 75 | 65 | 125 | 75 |
| $B$ | 90 | 78 | 66 | 132 | 78 |
| $C$ | 75 | 66 | 57 | 114 | 69 |
| $D$ | 80 | 72 | 60 | 120 | 72 |
| $E$ | 76 | 64 | 56 | 112 | 68 |

Is the solution unique ? If not, identify an alternative solution.
6. (a) Define pure and mixed strategy of a Game.
(b) Define saddle point of a game and value of a game.
(c) Reduce the following pay-off matrix $2 \times 2$ by dominance property and hence solve the problem :

|  | Player B |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Player A | 0 | 0 | 0 | 0 | 0 |  |
|  | 4 | 2 | 0 | 2 | 1 |  |
|  | 4 | 3 | 1 | 3 | 2 |  |
|  | 4 | 3 | 4 | -1 | 2 |  |

(d) Show that for a symmetric game the value of the game is zero.

## POINT SET TOPOLOGY

Answer any three questions.

1. (a) Define the cardinal number of a set.
(b) Define accumulation point or limit point.
(c) Let $\mathbb{N}$ be the set of all positive integers and $\tau$ be a topology on $\mathbb{N}$ consisting of $\phi$ and all the sets of the form $E_{n}:=\{n, n+1, n+2, \ldots\}$ where $n \in \mathbb{N}$. Then find the accumulation points of $A:=\{5,13,28,37\}$. Also determine those subsets of $\mathbb{N}$ whose derived set is $\mathbb{N}$.
(d) Let $\tau_{1}, \tau_{2}$ be two topologies on a non-empty set $X$ with $\tau_{1} \subseteq \tau_{2}$. Let $A \subseteq X$. Then prove that (i) every $\tau_{2}$ limit point of $A$ is a $\tau_{1}$-limit point of $A$; (ii) give an example to show that a $\tau_{1}$-limit point of $A$ need not be a $\tau_{2}$-limit point of $A$.
(e) Define finite intersection property. Then prove that a metric space $(X, d)$ is compact if and only if for every collection of closed sets $\left\{F_{\alpha}: \alpha \in A\right\}$ in $X$ possessing finite intersection property, the intersection $\bigcap_{\alpha \in \Lambda} F_{\alpha} \neq \phi$.
(b) Let X be a non-empty set. Define basis and sub-basis for a topology on $X$ with examples.
(c) Let $(X, \tau)$ and $(Y, U)$ be two topological spaces and $f: X \rightarrow Y$ be a mapping. Then prove that $f$ is continuous if and only if $F$ is closed subset in $Y \Rightarrow f^{-1}(F)$ is closed in $X$.
(d) Prove that a subset $S$ of $\mathbb{R}$ is connected if and only if $S$ is an interval.
(e) In a metric space $(X, d)$, prove that a subset $A$ of $X$ is compact $\Rightarrow A$ is totally bounded. Is the converse implication true? Give explanation in support of your answer.
2. (a) Prove that a countable union of countable sets is countable.
(b) Let $X$ be a non-empty set and $\tau_{1}, \tau_{2}$ be two topologies on $X$ with bases $B_{1}, B_{2}$, respectively. Show that if $\tau_{1} \subseteq \tau_{2}$, then for $G_{1} \in B_{1}$ and $x \in G_{1}$ there exists $G_{2} \in B_{2}$ such that $x \in G_{2} \subseteq G_{1}$.
(c) Let $(X, \tau)$ be a topological space and $A$ be a non-empty subset of $X$. Prove that (i) $A^{0}$ is the largest open set contained in $A$;
(ii) $A$ is open if and only if $A=A^{0}$, where $A^{0}$ denotes the interior of $A$.
(d) Let $Y$ be a subspace of $X$. Then show that $Y$ is compact if and only if every covering of $Y$ by sets open in $X$ contains a finite subcollection covering $Y$.
(e) Give an example of a topology on $\mathbb{R}$, other than the indiscrete one, with respect to which $\mathbb{R}$ becomes compact.
3. (a) Let $A$ and $B$ two well-ordered sets. Prove that $A \times B$ is well-ordered in the Dictionary ordering.
(b) Let $(X, \tau)$ be a topological space such that $A$ is a nowhere dense subset of $X$. Prove that $\bar{A}$ does not contain any non-emptry open set of $X$.
(c) Consider $\mathbb{R}$ with usual topology $\tau_{u}$. Prove that $\mathbb{Q}$ is not a connected subspace of $\left(\mathbb{R}, \tau_{u}\right)$.
(d) Prove that the continuous image of a connected space is connected.
(e) Prove that a function $f$ on a topological space to a product space is continuous if and only if the composition $\pi_{\alpha} \circ f$ is continuous for each projection $\pi_{\alpha}: \Pi_{\alpha \in \Lambda} X_{\alpha} \rightarrow X_{\alpha}$.
4. (a) Show that $\mathbb{Q}$ is countably infinite.
(b) In a topological space $(X, \tau)$, prove that a subset $A$ of $X$ is closed if and only if $b d(A) \subseteq A$ where $b d(A)$ denotes the boundary of $A$.
(c) Let $(X, \tau)$ be a topological space, $\left(Y, \tau_{Y}\right)$ be a subspace and $A \subseteq Y$. Then show that $\bar{A}_{Y}=\bar{A} \cap Y$ and $A^{0}=\left(A^{0}\right)_{Y} \cap Y^{0}$.
(d) Let $X_{n}$ be a metric space with metric $d_{n}$ for $n \in \mathbb{N}$. Then show that
$\rho(x, y)=\max \left\{d_{1}\left(x_{1}, y_{1}\right), d_{2}\left(x_{2}, y_{2}\right), \ldots, d_{n}\left(x_{n}, y_{n}\right)\right\}$
where $x=\left(x_{1}, x_{2}, \ldots, x_{n}\right)$ and $y=\left(y_{1}, y_{2}, \ldots, y_{n}\right)$, defines a metric on the product space $X_{1} \times X_{2} \times \ldots \times X_{n}$.3
(e) Prove that $\mathbb{R}$ with usual topology has a countable base.
(f) If $E$ is a connected subspace of a topological space $X$ and $F$ is a subset of $X$ such that $E \subseteq F \subseteq \bar{E}$, prove that $F$ is also a connected subspace of $X$.
5. (a) Prove that a finite product of countable sets is countable, using induction hypothesis.
(b) Show that the product topology on $\Pi_{\alpha \in \Lambda} X_{\alpha}$ is the coarsest topology relative to any topology on $\Pi_{\alpha \in \Lambda} X_{\alpha}$ where each projection $\pi_{\alpha}: \Pi_{\alpha \in \Lambda} X_{\alpha} \rightarrow X_{\alpha}$ is continuous. 4
(c) Prove that a topological space $X$ is connected if and only if the only subsets which are both open and closed in $X$ are $\phi$ and $X$ only.
(d) Show that a topological space $(X, \tau)$ is disconnected if and only if there exists a continuous mapping from $X$ onto the discrete two-point space $\{0,1\}$.
(e) Define locally compact topological space with an example.
(f) State the Lebesgue number lemma.

## THEORY OF EQUATIONS

Answer any three questions.

1. Use the method of synthetic division to find the quotient and remainder, when $x^{5}-4 x^{4}+8 x^{2}-1$ is divided by $(x-3)$. Find the condition so that $x^{3}+3 p x+q$ may have a factor of the form $(x-a)^{2}$. Express $x^{4}+5 x^{2}-3 x+2$ as a polynomial in $(x+2) .20$
2. Find the equation of fourth degree with rational coefficient, one root of which is $\sqrt{2}+\sqrt{3 i}$. State Descartes' rule of sign. Use it to determine the nature of roots of the equation $x^{n}-1=0$. Find the multiple root of the equation $x^{4}+2 x^{3}+2 x^{2}+2 x+1=0$.
3. Solve the equation $4 x^{3}+16 x^{2}-9 x-36=0$, when the sum of two roots is zero. Find the condition that the equation $x^{3}+p x^{2}+q x+r=0$ may have two equal roots but of opposite sign. If $m, n$ are integers prime to each other then prove that 1 is the only common root of the equations $x^{m}-1=0$ and $x^{n}-1=0$.
4. Find the equation whose roots are the squares of the roots of the equation $x^{3}+b x^{2}+c x+d=0$. Solve the cubic equation $x^{3}-18 x-35=0$ by Cardan's method. Find the relation between the coefficients of the equation $x^{3}+a x^{2}+b x+c=0$, when the roots $\alpha, \beta$ are connected by the relation $1+\alpha \beta=0$.
5. Solve the equation $x^{4}-x^{3}+2 x^{2}-x+1=0$ which has four distinct roots of equal moduli. Show that the equation $(x-a)^{3}+(x-b)^{3}+(x-c)^{3}+(x-d)^{3}=0$ where $a, b, c, d$ are positive and not all equal has only one real root. Remove the second term of the equation $x^{4}+4 x^{3}-7 x^{2}-22 x+24=0$ and hence solve the original equation.
6. Prove by Strum's theorem that the roots of the equation $x^{4}-4 x^{3}+2 x^{2}+4 x+1=0$ lie in the intervals $(-1,0)$ and $(2,3)$. Prove that the solution of any reciprocal equation depends on that of a reciprocal equation of first type and of even degree. If $\alpha, \beta, \mu$ be the roots of the equation $x^{3}+5 x^{2}+1=0$ find the value of $\sum \frac{1}{\alpha}$.

| বিদ্যাসাগর বিশ্ববিদ্যালয় <br> VIDYASAGAR UNIVERSITY <br> Question Paper |  |  |  |
| :---: | :---: | :---: | :---: |
| B.Sc. Honours Examinations 2021 <br> (Under CBCS Pattern) <br> Semester - V <br> Subject : MATHEMATICS <br> Paper : DSE 1-T |  |  |  |
| Full Marks : 60 Time : 3 Hours |  |  |  |
| Candidates are required to give their answers in their own words as far as practicable. <br> The figures in the margin indicate full marks. |  |  |  |
| [ LINEAR PROGRAMMING ] <br> Group-A <br> 1. Answer any four questions: <br> (i) (a) Prove that the set of all convex combinations of a finite number of linearly independent vectors $X_{l}, \ldots \ldots, X_{k}$ is a convex set. <br> (b) Solve the following L.P.P by Big-M method $\begin{array}{ll} \text { Maximize } & z=x_{1}-2 x_{2}+3 x_{3} \\ \text { Subject to } & x_{1}+2 x_{2}+3 x_{3}=15 \\ & 2 x_{1}+x_{2}+5 x_{3}=20 \\ & x_{1}, x_{2}, x_{3} \geq 0 \end{array}$ |  |  |  |

(ii) (a) Use graphical method, show that the following L.P.P have no feasible solution Maximize $\quad z=3 x+4 y$
Subject to $\quad x-y \leq 1$
$-x+y \leq 0$
$x, y \geq 0$
(b) Examine whether the set is convex or not

$$
X=\left\{\left(x_{1}, x_{2}\right), x_{1} \geq 2, x_{2} \leq 3, x_{1}, x_{2} \geq 0\right\}
$$

(iii) (a) Solve the following L.P.P using simplex method

Maximize $\quad z=x_{1}+2 x_{2}+x_{3}$
Subject to $\quad 2 x_{1}+x_{2}-x_{3} \geq-2$
$-2 x_{1}+x_{2}-5 x_{3} \leq 6$
$4 x_{1}+x_{2}+x_{3} \leq 6$
$x_{1}, x_{2}, x_{3} \geq 0$
(b) Write the dual of the following L.P.P.

Maximize $\quad z=x_{1}+2 x_{2}$
Subject to $\quad 2 x_{1}+3 x_{2} \geq 4$
$3 x_{1}+4 x_{2}=5$
$x_{1} \geq 0$ and $x_{2}$ is unrestricted.
(iv) (a) Prove that the set of all feasible solutions to a linear programming problem is a convex set.
(b) Use two phase simplex method to solve the following L.P.P

Maximize $\quad z=3 x_{1}+2 x_{2}$
Subject to $2 x_{1}+x_{2} \leq 40$
$x_{1}+x_{2} \leq 24$
$2 x_{1}+3 x_{2} \leq 60$
$x_{1}, x_{2} \geq 0$
$4+8$
(v) (a) Prove that the dual of the dual is primal.
(b) Solve the following transportation problem using North-West corner method:

|  | Destination |  |  |  | Supply |
| :---: | :---: | :---: | :---: | :---: | :---: |
|  | X | Y | Z | W |  |
| A | 5 | 4 | 6 | 14 | 15 |
| Origin B | 2 | 9 | 8 | 6 | 4 |
| C | 6 | 11 | 7 | 13 | 8 |
| Demand | 9 | 7 | 5 | 6 |  |

(vi) (a) Prove that the number of basis variables in a transportation problem with $m$ origin and n destinations is almost $m+n-1$.
(b) Solve the following travelling salesman problem :

|  | A | B | C | D | E |
| :--- | :---: | :---: | :---: | :---: | :---: |
|  | $\infty$ | 2 | 4 | 7 | 1 |
| B | 5 | $\infty$ | 2 | 8 | 2 |
| C | 7 | 6 | $\infty$ | 4 | 6 |
| D | 10 | 3 | 5 | $\infty$ | 4 |
| E | 1 | 2 | 2 | 8 | $\infty$ |
|  |  |  |  |  |  |

(vii) (a) Find the optimal assignment and the optimal assignment cost from the following cost matrix :

|  | $M_{1}$ | $M_{2}$ | $M_{3}$ | $M_{4}$ | $M_{5}$ |
| :--- | :---: | :---: | :---: | :---: | :---: |
| I | 9 | 8 | 7 | 6 | 4 |
| II | 5 | 7 | 5 | 6 | 8 |
| III | 8 | 7 | 6 | 3 | 5 |
| IV | 8 | 5 | 4 | 9 | 3 |
| V | 6 | 7 | 6 | 8 | 5 |
|  |  |  |  |  |  |

(b) Prove that if we add a fixed number to each element of a pay-of matrix, the optimal strategies remain unchanged but the value of the game is increased by that number.
(viii) (a) Solve graphically the game whose pay-off matrix is

> Player B

Player $A\left(\begin{array}{lllr}2 & 2 & 3 & -1 \\ 4 & 3 & 2 & 6\end{array}\right)$
(b) Use dominance property to reduce the pay-off matrix given by

Player B
Plaer $A\left(\begin{array}{rrrr}3 & -1 & 1 & 2 \\ -2 & 3 & 2 & 6 \\ 2 & -2 & -1 & 1\end{array}\right)$
into a $2 \times 2$ matrix and find the mixed strategies for A and B. Also find the value of the game.

## Group-B

2. Answer any six questions:
(i) Find the extreme point, if any, of the set

$$
S=\{(x, y):|x| \leq 1,|y| \leq 1\} .
$$

(ii) State the fundamental theorem of linear programming.
(iii) Define zero sum game.
(iv) Solve the games with the following payoff matrix $\left(\begin{array}{rr}6 & -3 \\ -3 & 0\end{array}\right)$.
(v) Prove that the solution of a transportation problem is never unbounded.
(vi) Define saddle point of a game.
(vii) Define basic feasible solution of an L.P.P.
(viii) For what values of a, the game with the following payoff matrix is strictly determinable?

|  |  | I | II |
| :--- | :---: | :---: | :---: |
| II |  |  |  |
| I | a | 5 | 2 |
| II | -1 | a | -8 |
| III | -2 | 3 | a |
|  |  |  |  |

(ix) Put the following problem in standard from

Maximize $\quad z=3 x_{1}-4 x_{2}-x_{3}$
Subject to $x_{1}+3 x_{2}-4 x_{3} \leq 12$

$$
\begin{aligned}
& 2 x_{1}-x_{2}+x_{3} \leq 20 \\
& x_{1}-4 x_{2}-5 x_{3} \geq 5 \\
& x_{1} \geq 0, x_{2} \text { and } \\
& x_{3} \text { are unrestricted in sign. }
\end{aligned}
$$

(x) In a game with the $2 \times 2$ payof matrix

$$
\left(\begin{array}{ll}
a & b \\
c & d
\end{array}\right)
$$

where $a<d<b<c$, show that there is no saddle point.

## OR

## [ POINT SET TOPOLOGY ]

## Group-A

1. Answer any four questions :
(a) (i) Check whether the set $A=\left\{f:\{0,1\} \rightarrow \mathbb{Z}^{+} \mid f\right.$ is a function and $\mathrm{Z}^{+}$denote the set of all positive integers $\}$ is countable or not.
(ii) Prove $\left(\mathbb{R}, \tau_{\mathrm{u}}\right)$, where $\tau_{\mathrm{u}}$ denotes the usual topology on $\mathbb{R}$, has a countable base.
(iii) Show that $S=\{(-\infty, a) \mid a \in \mathbb{R}\} \cup\{(b, \infty) \mid b \in \mathbb{R}\}$ is a subbase for the lower limit topology $\tau_{l}$ on $\mathbb{R}$.
(iv) Prove that the union of collection of connected sets having a point in common is connected.
(b) (i) Show that every well-ordered set has the least upper bound property.
(ii) Prove that in a topological space $(X, \tau), b d(A)=\bar{A} \cap \bar{A}^{C}$ where $A$ is a nonempty subset of $X$.
(iii) Let $(X, \tau),\left(Y, \tau^{\prime}\right)$ be two topological spaces and $b \in Y$. Then prove that $X$ and $X \times\{b\}$ are homeomorphic. Hence prove that the product space $X \times Y$ is connected, if $X$ and $Y$ are connected.
(iv) Prove that $\mathbb{R}$ with respect to the usual topology $\tau_{\mathrm{u}}$ is not compact.

$$
2+2+(3+3)+2=12
$$

(c) (i) Let $A_{1}$ and $A_{2}$ be disjoint sets, well-ordered by $<_{1}$ and $<_{2}$, respectively. Define an order relation on $A_{1} \cup A_{2}$ by letting $a<b$ either if $a, b \in A_{1}$ and $a<1 b$, or if $a, b \in A_{2}$ and $a<_{2} b$, or if $a \in A_{1}$ and $b \in A_{2}$. Show that this is a wellordering.
(ii) Consider a family of non-empty sets $\left\{X_{\alpha} \mid \alpha \in \Lambda\right\}$ where $\Lambda$ is an infinite set. Prove that the box topology is finer than the product topology on $\Pi_{\alpha \in \Lambda} X_{\alpha}$.
(iii) Define totally bounded metric space with an example.
(d) (i) Given two points $\left(x_{0}, y_{0}\right)$ and $\left(x_{1}, y_{1}\right)$ in $\mathbb{R}^{2}$, define $\left(x_{0}, y_{0}\right)<\left(x_{1}, y_{1}\right)$ if $x_{0}<x_{1}$ and $y_{0} \leq y_{1}$. Show that the curve $y=x^{3}$ is a maximal simply ordered subset of $\mathbb{R}^{2}$.
(ii) Let $X$ be a non-empty set and consider the following metric on X :
$d(x, y)= \begin{cases}1 & \text { if } \\ 0 & \text { if } \\ x=y\end{cases}$
Prove that the metric topolgocy on $X$ induced by $d$ is the discrete topology on $X$.
(iii) Define open map, Let $X$ be the subspace $[0,1] \cup[2,3]$ of the topological space $\left(\mathbb{R}, \tau_{\mathrm{u}}\right)$ and $Y$ be the subspace $[0,2]$ of the topological space $\left(\mathbb{R}, \tau_{\mathrm{u}}\right)$ where $\tau_{\mathrm{u}}$ denotes the usual topology on $\mathbb{R}$ ). Is the map $p: X \rightarrow Y$ an open map where $p$ is defined as follows :
$p(x)=\left\{\begin{array}{c}x, \text { if } x \in[0,1] \\ x-1, \text { if } x \in[2,3] \text { ? }\end{array}\right.$
Justify your answer.
(iv) Prove that every closed subspace of a compact space is compact.

$$
3+2+(1+2)+4=12
$$

(e) (i) Using the Axiom of choice show that if $f: A \rightarrow B$ is a surjective function, then f has a right inverse $h: B \rightarrow A$.
(ii) Deine a quotient map between two topological spaces. Then deine the quotient topology induced by the function $p$ on a set $A$ where $(X, \tau)$ is a topologcial space and $p: X \rightarrow A$ is a surjective map.
(iii) Consider $\left(\mathbb{R}, \tau_{\mathrm{u}}\right)$ where $\tau_{\mathrm{u}}$ denote the usual topology on $\mathbb{R}$. Define a surjective $\operatorname{map} p$ from $\mathbb{R}$ onto a three-element set $A=\{a, b, c\}$ defined by

$$
p(x)=\left\{\begin{array}{l}
a, \text { if } x>0 \\
b, \text { if } x<0 \\
c, \text { if } x=0
\end{array}\right.
$$

Compute the quotient topology on $A$ induced by the surjective function $p$.
(iv) Deine finite complement (co-finite) topology on a set $X$. Then show that an infinite set $X$ is always connected with respect to the finite complement topology.
(v) Give example (with justification) of a locally connected subspace which is not a connected subspace in $\left(\mathbb{R}, \tau_{\mathrm{u}}\right)$ where $\tau_{\mathrm{u}}$ denotes the usual topology on $\mathbb{R}$. $2+(1+1)+2+(1+2)+3=12$
(f) (i) Consider the strict partial orderd set $\left(\mathbb{R}^{2},<\right)$ where for $\left(x_{0}, y_{0}\right),\left(x_{1}, y_{1}\right) \in \mathbb{R}^{2}$, $\left(x_{0}, y_{0}\right)<\left(x_{1}, y_{1}\right)$ if any only if $x_{0}=y_{1}$ and $x_{0}<x_{1}$. Find a maximal simple ordered subset of $\left(\mathbb{R}^{2},<\right)$.
(ii) Let $(X, \tau)$ and $\left(X, \tau^{\prime}\right)$ be two topological spaces, $B \subseteq \tau^{\prime}$ be a base for the topology $\tau^{\prime}$ on $Y$ and $f: X \rightarrow Y$ be a mapping. Then prove that $f$ is continuous if and only $f^{-1}(B) \in \tau$ for all $B \in \mathrm{~B}$.
(iii) Let $\tau$ and $\tau^{\prime}$ be two topologies on a set $X$ such that $\tau \subseteq \tau^{\prime}$. Prove that the connectedness of $\left(X, \tau^{\prime}\right)$ implies connectedness of $(X, \tau)$. By exhibiting an example (with justification) show that the connectedness of $(X, \tau)$ need not always imply the connectedness of $\left(X, \tau^{\prime}\right)$.
(g) (i) State Schroeder-Bernstein Theorem.
(ii) Give example of a continuous bijective function between two topological spaces which fails to be a homeomorphism.
(iii) Consider a family of topological spaces $\left\{\left(X_{a}, \tau_{a}\right) \mid \alpha \in \Lambda\right\}$ where $\Lambda$ is an infinite set. Let $A_{\alpha} \subseteq X_{\alpha}$ for each $\alpha \in A$. Then prove that $\overline{\prod_{\alpha \in \Lambda} A_{\alpha}}=\prod_{\alpha \in \Lambda} \overline{A_{\alpha}}$ holds in the product topology on $\prod_{\alpha \in \Lambda} X_{\alpha}$.
(iv) Show that finite union of compact subspaces in a topological space in compact again.
$2+3+4+3=12$
(h) (i) Show that in a well-ordered set, every element except the largest (if exists) has an immediate successor.
(ii) Give an example (with justification) of a function on a topological space which is continuous precisely at one point.
(iii) Prove that the image of a compact space under a continuous map is compact.
(iv) Let $\tau_{1}$ and $\tau_{2}$ be two topologies on a set $X$ such that $\tau_{1} \subseteq \tau_{2}$. Does the compactness of $\left(X, \tau_{1}\right)$ imply the compactness of $\left(X, \tau_{2}\right)$ ? Justify your answer explicitly.

## Group-B

2. Answer any six questions :
(a) Give an example (with justiication) of a countably infinite set.
(b) State axiom of choice.
(c) Prove that $(0,1) \subseteq \mathbb{R}$ is not well ordered.
(d) Show that any uncountable set has greater cardinality than $\mathbb{N}$.
(e) Prove that the usual topology is coarser than the lower limit topology on $\mathbb{R}$.
(f) Consider the set $Y=[-1,1]$ as a subspace of $\mathbb{R}$ with the usual topology. Is $A=\left\{x \in \mathbb{R}\left|\frac{1}{2}<|x|<1\right\}\right.$ open in $Y$ ? Justify your answer.
(g) Consider a family of topological spaces $\left\{\left(X_{\alpha}, \tau_{\alpha}\right) \mid \alpha \in \Lambda\right\}$ and the Cartesian product $\Pi_{\alpha \in \Lambda} X_{\alpha}$. Define product topology on $\Pi_{\alpha \in \Lambda} X_{\alpha}$.
(h) Let $(X, \tau)$ be a topological space, $A \subseteq X$ and $\tau_{A}$ denotes the subspace topology. Prove that the inclusion map from $\left(A, \tau_{A}\right)$ into $(X, \tau)$ is a continuous map.
(i) Define path connected topological space with an example.
(j) Consider the subspace $A=\{0\} \cup\left\{\left.\frac{1}{n} \right\rvert\, n \in \mathbb{N}\right\}$ in the topological space $\mathbb{R}$ with the usual topology $\tau_{u}$. Prove that $A$ is compact.

## OR

## [ THEORY OF EQUATION ]

## Group-A

Answer any four questions :

1. (a) For what integral value $m, x^{2}+x+1$ is a factor of $x^{2 m} \_x^{m}+1$ ?
(b) If $f(x)$ be a polynomial in $x$ of degree $n$ and $\alpha$ is any number real or complex, then show that

$$
f(x)=f(\alpha)+f^{\prime}(\alpha)(x-\alpha)+f^{\prime \prime}(\alpha)(x-\alpha)^{2}+\ldots \ldots+f^{n}(\alpha)(x-\alpha)^{n}
$$

(c) If $x^{4}+p x^{2}+q x+r$ has a factor of the form $(x-a)^{3}$, then show that $8 p^{3}+27 q^{2}=0$ and $p^{2}+12 r=0$.
2. (a) A polynomial $f(x)$ leaves a remainder 10 when it is divided by $(x-2)$ and the reminder $(2 x-3)$ when it is divided by $(x+1)^{2}$. Find the remainder when it is divided by $(x-2)(x+1)^{2}$.
(b) If $f(x)$ be a polynomial in $x$ and $a, b$ are unequal, show that remainder in the division of $f(x)$ by $(x-a)(x-b)$ is $\frac{(x-b) f(a)-(x-a) f(b)}{(a-b)}$.

$$
4+4+4=12
$$

3. (a) Solve the equation $x^{4}-x^{3}+2 x^{2}-x+1=0$, which has four distinct roots of equal moduli.
(b) Find the conditions for which the equation $x^{4}-14 x^{3}+24 x+k=0$, has (i) four unequal real roots, (ii) two distinct real roots, (iii) no real root.
(c) Let $f(x)=a_{0} x^{n}+a_{1} x^{n-1}+\ldots . .+a_{n}$ where $a_{0}, a_{1} \ldots \ldots a_{n}$ are integers. If $f(0), f(1)$ be both odd prove that the equation $f(x)=0$ cannot have an integer root.

$$
4+4+4=12
$$

4. (a) Use Sturm's theorem to find nature and position of the real roots of the equation $x^{3}-7 x+7=0$.
(b) Prove that the solution of any reciprocal equation depends on that of a reciprocal equation of first type and of even degree.
(c) If $\alpha, \beta, \gamma, \delta$ be the roots of the equation $x^{4}+4 p x^{3}+6 q x^{2}+4 r x+s=0$ find the value of $\sum \alpha^{2} \beta^{2}(\gamma-\delta)^{2}$. $4+4+4=12$
5. (a) Let $\alpha_{1}, \alpha_{2} \ldots \ldots . \alpha_{n}$ be the roots of the equation $f(x)=x^{n}+p_{1} x^{n-1}+p_{2} x^{n-2}+\ldots . .+p_{n}=0 \quad$ and $\quad$ let $\quad S_{r}=\alpha_{1}^{r}+\alpha_{2}^{r}+\alpha_{3}^{r}+\ldots . . .+\alpha_{n}^{r}$, where $r>0$, is an intreger. Then show that-
(i) $S_{r}+p_{1} S_{r-1}+\ldots .+p_{r-1} S_{1}+r p_{r}=0$ if $12 \leq r<n$
(ii) $S_{r}+p_{1} S_{r-1}+\ldots . .+p_{n} S_{r-n}=0$ if $r>n$.
(b) If $\alpha, \beta, \gamma, \delta$ be the roots of the equation $x^{4}+p x^{3}+q x^{2}+r x+s=0, s \neq 0$ then find the values of $\sum \frac{\alpha \beta}{\gamma}, \sum \frac{\alpha^{2}}{\beta}$.
6. (a) Show that the equation $x^{3}-16 x^{2}-x-1=0$ has only positive root.
(b) If equation $f(x)=0$ has all roots real, then show that the equation $f f^{\prime \prime}-\left\{f^{\prime}\right\}^{2}=0$ has all its root imaginary.
(c) If the roots of the equation $x^{3}+3 p x^{2}+3 q x+r=0$ are in H.P., then show that

$$
2 q^{3}=3 p q r-r^{2} .
$$

$$
4+4+4=12
$$

7. (a) Find the substitution of the form $x=m y+n$ which will transform the following equation to a reciprocal one and hence solve it $x^{4}-7 x^{3}+13 x^{2}-12 x+6=0$.
(b) Solve the equation $x^{5}-1=0$, Hence find the value of $\cos \frac{\pi}{5}, \cos \frac{2 \pi}{5} . \quad 8+4=12$
8. (a) Solve the equation $x^{7}-1=0$. Deduce that $2 \cos \frac{2 \pi}{7}, 2 \cos \frac{4 \pi}{7}, 2 \cos \frac{8 \pi}{7}$ are the roots of the equation $t^{3}+t^{2}-2 t-1=0$.
(b) Solve the equation $x^{3}-13 x-35=0$, by taking $x=u+v$.

## Group-B

9. Answer any six questions :
(a) Prove that the roots of the equation $(2 x+3)(2 x+4)(x-1)(4 x-7)+$ $(x+1)(2 x-1)(2 x-3)=0$ are all real and different.
(b) If $\alpha, \beta, \gamma, \delta$ be the roots of the equation $x^{4}-x^{3}+2 x^{2}+x+1=0$, find the value of $\left(\alpha^{3}+1\right)\left(\beta^{3}+1\right)\left(\gamma^{3}+1\right)\left(\delta^{3}+1\right)$.
(c) Express $x^{5}-5 x^{4}+12 x^{2}-1$ as a polynomial in $(x-1)$.
(d) State Descartes rule of signs.
(e) Apply Descartes rule of signs to find the nature of the roots of the equation $x^{6}+7 x^{4}+x^{2}+2 x+1=0$.
(f) If $\alpha$ be amultiple root of order 3 of the equation $x^{4}+b x^{2}+c x+d=0(d \neq 0)$, show that $\alpha=-\frac{8 d}{3 c}$
(g) If $(x)=x^{4}-3 x^{2}+10 x$, express $f(x+3)$ as a polynomial in x .
(h) Determine the multiple roots of the equation $x^{5}+2 x^{4}+2 x^{3}+4 x^{2}+x+2=0$.
(i) Solve the equation $x^{4}+x^{2}-2 x+6=0$, it is given that $1+i$ is a root.
(j) How many times the graph of the polynomial $\left(x^{3}-1\right)\left(x^{2}+x+1\right)$ will cross x axis?

10. (a) Prove that for any random variable $X$ (discrete or continuous) and for any real number $c E(|X-c|) \geq E(|X-\mu|)$, Provided the expectations exists and $\mu$ is the medial of $X$.
(b) Let $X$ be a random variable having Poisson distribution with parameter $\mu$ and the conditional distribution of $Y$ given $X=i$ be given by $f_{i, j}=\binom{i}{j} p^{i} q^{j}$ for $0 \leq j \leq i$, $i \neq 0, p+q=1$. Find the marginal distribution of $Y$.
(c) If $f(x, y)=\left\{\begin{array}{cc}\frac{6-x-y}{8}, 0<x<2,2<y<4 \\ 0, & \text { elsewhere }\end{array}\right.$, find $P(X+Y<3)$.
11. (a) The jdf (joint density function) of $X$ and $Y$ is given by
$f(x, y)=\left\{\begin{array}{cc}k(x+y), & 0<x<10<y<1 \\ 0, & \text { elsewhere }\end{array}\right.$

Find find $P(|X-Y|) \leq 1 / 2$ and $f_{X}(x)$ and $f_{Y}(y)$. Are $X$ and $Y$ independent?
(b) Let $X$ and $Y$ be dindependent random variable having the normal density $(0, \sigma)$. Find $P\left(x^{2}+y^{2} \leq 1\right)$.
(c) The joint probability density function of the random variable $X$ and $Y$ is
$f(x, y)=\left\{\begin{array}{cl}k(1-x-y), & x \geq 0, y \geq 0, x+y \leq 1 \\ 0, & \text { elsewhere }\end{array}\right.$,
where $k$ is a constant. Find mean value of $Y$ when $X=1 / 2$ and the covariance of $X$ and $Y$.
4. (a) If $X$ and $Y$ are connected by $2 X+3 Y+4=0$, then show that $\rho(X, Y)=-1$. 6
(b) Let the joint probability density function of $X$ and $Y$ be given by $f(x, y)=x^{2}+\frac{x y}{3}, 0, x, 1,0, y, 2: 0$ elsewhere. Find regression line of $x$ on $y$. 7
(c) If $X$ and $Y$ are two independent random variable having the density function respectively
$f_{X}(x)=\left\{\begin{array}{ll}e^{-x}, & x>0 \\ 0 & \text { elsewhere }\end{array}\right.$ and $f_{Y}(y)= \begin{cases}e^{-y}, & y>0 \\ 0 & \text { elsewhere }\end{cases}$

Find the density function of $\frac{X}{X+Y}$.
5. (a) Show by Chebyshev's inequality that 2000 throws with a coin the probability that the number of heads lies between 900 to 1100 is 19/20.
(b) A random variable $X$ has probability density function $12 x^{2}(1-x),(0<x<1)$.

Compute $P(|x-m| \geq 2 \sigma)$, compare it with the limit given by Chebyshev's inequality.
(c) A random sample of 500 apples was taken from a large consignment and 60 were bad. Obtain the $98 \%$ confidence limits for the percentage number of bad apples in the consignment.
6. (a) Sample of two types of electric light bulb were tested for length of life and the following data were obtained:

|  | Type-I | Type-II |
| :--- | :--- | :--- |
| Sample no | $n_{1}=8$ | $n_{2}=7$ |
| Sample means | $\bar{x}_{1}=1234 \mathrm{hrs}$ | $\bar{x}_{2}=1036 \mathrm{hrs}$ |
| Sample s.d | $s_{1}=36 \mathrm{hrs}$ | $s_{2}=40 \mathrm{hrs}$ |

Is the difference in the mean sufficient to warren that the Type I is superior to Type II regarding the length of life.
(b) Obtain the recurrence relation $\mu_{K+1}=\mu\left(K \mu_{K-1}+\frac{d \mu_{K}}{d \mu}\right)$ for the Poisson distribution with parameter $\mu$. Hence, find the coefficient of Skewness and Coefficient of excess of this Poisson distribution.
(c) If $X$ is uniformly distribution over $(-1,1)$, then find the distribution of $|X|$.

## BOOLEAN ALGEBRA AND AUTOMATA THEORY

Answer any three questions.

1. (a) Tabulate the Chomsky hierarchy with an example for each type of grammar.
(b) What are universal logic gate? Why those are called universal?
(c) With a suitable example, explain various asymptotic notations.
(d) Explain lattice, sublattice, explain with example.
2. Construct a Turing Machine that recognizes the language $L=\left\{0^{n m}: n, m>=0\right\}$.
3. Reduce the given CFG with Productions given by $\mathrm{S} \rightarrow \mathrm{abSB} / \mathrm{a} / \mathrm{aAb}$ and
$\mathrm{A} \rightarrow \mathrm{bS} / \mathrm{aAAb}$ to Chomsky Normal form.
4. Deduce R.E. from the Fig. and check whether the string 0100 is accepted or not.

5. Define a regular set. Using Pumping Lemma, show that the language $L=\left\{a^{n} b^{k}: n>k\right.$ and $\left.n>=1\right\}$ is not regular.
6. Among the first 1000 positive integers :
(a) Determine the integers which are not divisible by 5 , nor by 7 nor by 9 .
(b) Determine the integers divisible by 5 but not by 7 not by 9 .

## PORTFOLIO OPTIMIZATION

Answer any three questions.

1. Prove that the expected return $\mu_{i}$ on any asset $i$ satisfies $\mu_{i}=r_{f}+\beta_{i}\left(\mu_{M}-r_{f}\right), \beta_{i}=\frac{\sigma_{i M}}{\sigma_{M^{2}}}$ and $\sigma_{i M}$ is the covariance of the return on asset $i$ and the market protfolio $r_{M}$; $\sigma_{M}^{2}=\operatorname{var}\left(r_{M}\right)$.
2. Consider 3 assets with rates of return $r_{1}, r_{2}$ and $r_{3}$ respectively. The covariance matrix and expected rates of return are $\Sigma=\left(\begin{array}{lll}2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2\end{array}\right)$ and $m=\left(\begin{array}{l}0.4 \\ 0.4 \\ 0.8\end{array}\right)$
(a) Find the minimum variance portfolio.
(b) Find a second efficient portfolio.
(c) If the risk free rate is $r_{f}=0.2$, find an efficient portfolio of risky assets.
3. For the Markowitz mean-variance portfolio, solve the quadratic programming problem

Minimize $\quad \frac{1}{2} w^{T} \Sigma w-\lambda m^{T} w$

Subject to $e^{T} w=1$
where $w=\left(w_{1}, w_{2}, \ldots \ldots w_{n}\right)^{T}, m=\left(m_{1}, m_{2}, \ldots \ldots m_{n}\right)^{T}$
$\mu_{i}=E\left(r_{i}\right), \quad z=\left(r_{1}, r_{2}, \ldots \ldots r_{n}\right)^{T}, \operatorname{cov}(z)=\Sigma$
4. Assume that the expected rate of return on the market portfolio is $24 \%\left(r_{M}=0.24\right)$ and the rate of return on T-Bills (risk free rate) is $7 \%\left(r_{f}=0.07\right)$. The standard deviation of the market is $33 \%\left(\sigma_{M}=0.33\right)$. Assume that the market portfolio is efficient.
(a) What is the equation for the capital market line?
(b) If an expected return of $38 \%$ is desired, what is the standard deviation of this position?
5. (a) Define (i) Beta of a portfolio
(ii) Security market line
(b) You have a protfolio with a beta of 0.84 . What will be the new portfolio beta if you keep $85 \%$ of your money in the old portfolio and $14 \%$ in a stock with a beta of 1.93 ?
6. (a) What are some of the benefits of diversifiction ?
(b) Use the information in the following to answer the questions below.

| State of Economy | Probability of state | Return on A in state | Return on B in state |
| :--- | :--- | :--- | :--- |
| Boom | $35 \%$ | 0.040 | 0.210 |
| Normal | $50 \%$ | 0.030 | 0.080 |
| Recession | $15 \%$ | 0.046 | -0.010 |

(i) What is the expected return of each asset ?
(ii) What is the variance of each asset ?

## 5th Semester Examination MATHEMATICS (Honours)

## Paper : C 11-T

[Partial Differential Equations and Applications]
[CBCS]
Full Marks : 60
Time : Three Hours
The figures in the margin indicate full marks.
Candidates are required to give their answers in their own words as far as practicable.

Group - A

1. Answer any ten questions: $2 \times 10=20$
(a) What is ballistics? Write different types of ballistics.
(b) Define quasi-linear and semi-linear partial differential equation.
(c) Find the general solution of second order PDE

$$
4 \frac{\partial^{2} z}{\partial x^{2}}-\frac{\partial^{2} z}{\partial y^{2}}=0
$$

(d) What is the nature of the second order PDE

$$
\frac{\partial^{2} z}{\partial y^{2}}-y \frac{\partial^{2} z}{\partial x^{2}}+x^{3} z=0 ?
$$

(e) Let $a, b \in \mathbb{R}$ be such that $a^{2}+b^{2} \neq 0$. Then prove that the Cauchy problem $a \frac{\partial z}{\partial x}+b \frac{\partial z}{\partial y}=1$, $x, y \in \mathbb{R}$ with $z(x, y)=x$ on $a x+b y=1$ has a unique solution.
(f) Find the characteristic curve of PDE: $2 y \frac{\partial z}{\partial x}+\left(2 x+y^{2}\right) \frac{\partial z}{\partial y}=0$ which is passing through the point $(0,0)$.
(g) Find the equations of the characteristic curves of the PDE $\left(x^{2}+2 y\right) \frac{\partial^{2} z}{\partial x^{2}}+\left(y^{3}-y+x\right) \frac{\partial^{2} z}{\partial y^{2}}$ $+x^{2}(y-1) \frac{\partial^{2} z}{\partial x \partial y}+3 \frac{\partial z}{\partial x}+z=0$ which are passing through the point $x=1, y=1$.
(h) Let $z(x, t)$ be the equation of $\frac{\partial^{2} z}{\partial x^{2}}=\frac{\partial^{2} z}{\partial t^{2}}$ with $z(x, 0)=\cos (5 \pi x)$ and $\frac{\partial z}{\partial t}(x, 0)=0$. Then prove that $z(1,1)=1$.
(i) Show that the solution of the PDE: $x \frac{\partial z}{\partial x}+y \frac{\partial z}{\partial y}=0$ is of the form $f(y / x)$.
(j) Prove that the partial differential equation $x \frac{\partial^{2} z}{\partial x^{2}}+y \frac{\partial^{2} z}{\partial y^{2}}=0$ is elliptic type for $x<0, y>0$.
(k) Find the two families of surfaces that generate the characteristics of $(3 y-2 z) \frac{\partial z}{\partial x}+(z-3 x) \frac{\partial z}{\partial x}$ $=2 x-y$.
a) Find the partial differential equation by eliminating arbitrary constants $a$ and $b$ from $z=(x+a)(y+b)$.
(m) Define apsidal angle and apsidal distance.
(a) Prove that a planet has only a radial acceleration towards the Sun.
(9) Prove that at an apse on a central orbit, the velocity is proportional to the reciprocal of the radius vector.

## Group - B

## 2. Answer any four questions :

(a) A particle moves with a central acceleration $\mu \div$ (distance) $)^{2}$; it is projected with velocity $v$ at a distance $\mathbb{R}$. Show that its path is a rectangular hyperbola if the angle of projection is $\sin ^{-1}\left[\mu /\left\{V R\left(V^{2}-\frac{2 \mu}{R}\right)^{1 / 2}\right\}\right]$.
P.T.O.

## ( 4 )

(b) A spherical raindrop falls through a cloud while accumulating mass at a rate $\lambda r^{2}$ where $r$ is its radius and $\lambda>0$. Find its velocity at time $t$ if it starts from rest with radius $a$.
(c) Find the integral surface of the PDE, $x(z+2 a) p$

$$
+(x z+2 y z+2 a y) q=z(z+a) .
$$

(d) Using the method of separation of variables, solve : $\frac{\partial z}{\partial x}=q \frac{\partial z}{\partial t}+z$ where $z(x, 0)=6 e^{-3 x}$.
(e) Reduce the wave equation $\frac{\partial^{2} z}{\partial t^{2}}=c^{2} \frac{\partial^{2} z}{\partial x^{2}}$ to canonical form.
(f) Solve $z^{2}=p q x y$ by Charpit's method.

## Group - C

3. Answer any two questions:
$10 \times 2=20$
(a) (i) If a point moves on a Curve with constant tangential acceleration such that the magnitudes of the tangential velocity and normal acceleration are in a constant ratio, find the $(s, \psi)$ equation of the curve.
(ii) Solve $\left(D^{3}-3 D D^{\prime 2}-2 D^{\prime 3}\right) z=\cos (x+2 y)$.
(b) (i) A particle is projected with velocity $V$ from
the cusp of a smooth inverted cycloid down the arc, show that the time of reaching the
vertex is $2 \sqrt{\frac{a}{g}} \tan ^{-1}\left[\sqrt{\frac{4 a g}{V}}\right]$.
(ii) Using the method of separation of variables, solve the following wave equation described by
$\mathrm{PDE}: \frac{\partial^{2} z}{\partial t^{2}}=4 \frac{\partial^{2} z}{\partial x^{2}}$
$\mathrm{BCS}: z(0, t)=0, z(s, t)=0$
ICS : $z(x, 0)=0,\left(\frac{\partial z}{\partial t}\right)_{t=0}=5 \sin \pi x$. $5+5$
(c) (i) Solve the boundary value problem $\frac{\partial u}{\partial t}=k \frac{\partial^{2} u}{\partial x^{2}}, \quad u(x, 0)=20, \quad u(0, t)=0$. $u(L, t)=0$.
(ii) Find the integral surface of the linear PDE $2 y(z-3) \frac{\partial z}{\partial x}+(2 x-z) \frac{\partial z}{\partial y}=y(2 x-3)$ which passes through the circle $x^{2}+y^{2}=2 x, z=0$. $5+5$

0
i. PTO.

## ( 6 )

(d) (i) Solve two dimensional Laplace's equation $\frac{\partial^{2} z}{\partial x^{2}}+\frac{\partial^{2} z}{\partial y^{2}}=0$
with $\operatorname{BCS} z(0, y)=0, z(l, y)=0$
and $z(x, y) \rightarrow 0$ as $y \rightarrow \infty$

$$
z(x, 0)=f(x)
$$

(ii) Let $u(x, y)$ be the solution of the Cauchy problem $\quad \frac{\partial u}{\partial y}-x \frac{\partial u}{\partial x}+u=1, \quad$ where
$-\infty<x<\infty, y \geq 0$ and $u(x, 0)=\sin x$, then find $u(0,1)$.

# 5th Semester Examination MATHEMATICS (Honours) 

## Paper : C 12-T

[Group Theory II]
[CBCS]
Full Marks : 60
The figures in the margin indicate full marks.
Candidates are required to give their answers in their own words as far as practicable.

## Group - A

1. Attempt any ten questions:
(a) Find two non-isomorphic groups $H_{1}$ and $H_{2}$ such that $\operatorname{Aut}\left(H_{1}\right)$ is isomorphic with $\operatorname{Aut}\left(H_{2}\right)$.
(b) Let $G$ be a group. Then prove that $|\operatorname{Inn}(G)|=1$ if and only if $G$ is commutative.
(c) Verify whether $\mathbb{Z}_{2} \oplus \mathbb{Z}_{2}$ is isomorphic to $\mathbb{Z}_{4} \cdot(0,5,2,3)$
(d) Let $G$ be a cyclic group of order 2023. Find the number of automorphisms defined on $G$.
P.T.O.
(e) Express $U(165)$ as an external direct product of cyclic groups of the form $\mathbb{Z}_{n}$.
(f) Give an example of a group $G$ such that $|G|=12$ and $G$ has more than one subgroup of order 6 .
(g) Define characteristic subgroup of a group $G$. Is it true that every normal subgroup is characteristic?

- Give reasons in support of your answer.
(h) Find the class equation for the Klein's four group.
(i) Let $p, q$ be odd primes and let $m$ and $n$ be positive integers. Is $U\left(p^{m}\right) \times U\left(q^{n}\right)$ cyclic? Justify your answer. Here $U(n)$ denotes the group of units modulo $n$.
(j) Let $G$ be a $p$-group (where $p$ is a prime) and $H$ be a non-trivial homomorphic image of $G$. Then prove that $H$ is also a $p$-group.
(k) Let $R$ denote the set of all polynomials with integer coefficients in the independent variables $x_{1}, x_{2}, x_{3}$. Let $S_{3}$ act on $R$ by $\sigma \cdot p\left(x_{1}, x_{2}, x_{3}\right)=$ $p\left(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}\right)$. Find the stabilizer of the polynomial $x_{1} x_{2}$ under the action of $G$.
(l) Express the Klein's four group as an internal direct product of two of its proper subgroups.

$$
(3)
$$

(iii) Find the conjugacy classes of $c l((1,2))$ and cl $((1,2,3))$ in $S_{3}$.
(n) Verify whether a non-commutative group of order 343 is simple.
(o) State fundamental theorem for finite abelian groups:

## Group - B

2. Attempt any four questions :
$5 \times 4=20$
(a) Prove that commutator subgroup $G^{\prime}$ of a group $G$ is a characteristic subgroup of $G$.
(b) Let $G$ be a group. Define commutator subgroup of $G$. Prove that Commutator subgroup $G^{\prime}$ is a normal subgroup of $G$ and $G / G^{\prime}$ is commutative.
$1+4$
(c) Find all subgroups of order 3 in $\mathbb{Z}_{9} \oplus \mathbb{Z}_{3}$.
(d) Determine all non-isomorphic abelian groups of order 720.
(e) Let $G$ be a group of order 60. If Sylow 3 -subgroup is normal in $G$ then show that Sylow 5 -subgroup is also normal in $G$.
(f) Let $G$ be a group. Prove that the mapping $\phi: G \times G \rightarrow G$ defined by $\phi(g, a)=g \cdot a=g a g^{-1}$ is a group action. Find its kernel and stabilizer $G_{a}$.

## Group - C

3. Attempt any two questions:
$10 \times 2=20$
(a) (i) Find $A u t(\mathbb{Z})$.
(ii) If $G$ is a non-abelian group then show that Aut $(G)$ can not be cyclic. $\quad P(a) f(b)=f(b) P(a)$
(iii) Prove that $\operatorname{Inn}(G) \approx \frac{G}{Z(G)}$, where $\operatorname{Inn}(G)$ is the group of inner automorphism of $G$ and $Z(G)$ is the centre of $G$.
(b) (i) Show that $\mathbb{Z}_{8} \oplus \mathbb{Z}_{2}$ is not isomorphic to $\mathbb{Z}_{4} \oplus \mathbb{Z}_{4}$.
(ii) Find all conjugacy classes of the Dihedral group $D_{8}$ of order 8 and hence verify the class equation.
(iii) Let $G$ and $H$ be finite cyclic groups. Prove that $G \oplus H$ is cyclic if and only if $|G|$ and $|H|$ are relatively prime.
(c) (i) State Cauchy's theorem. Use Cauchy's theorem to prove that if a finite group $G$ is a $p$-group then $|G|=p^{n}$ for some positive integer $n$, where $p$ is a prime.
(ii) Let $G$ be a group of order $p n$, where $p$ is a prime and $p>n$. Show that there exists a subgroup of order $p$ in $G$ which is normal.
(iii) Find the number of elements of order 5 in the direct product $\mathbb{Z}_{25} \oplus \mathbb{Z}_{5}$. $3+3+4$
(d) (i) Let $G$ be a group acting on a non-empty set $S$ and $a \in S$. Then prove that $\| a]=\left[G: G_{a}\right]$ where $[a]$ denotes the orbit of $a$ and $G_{a}^{a}$ denotes the stabilizer of $a$.
(ii) Let $G$ be a finite group and $H$ be a proper subgroup of $G$ with index $n$ such that $|G|$ does not divide $n$ !. Using group action show that $G$ contains a non-trivial normal subgroup. Hence show that a simple group of order 63 cannot contain a subgroup of order 21.
$4+(3+3)$

# 5th Semester Examination MATHEMATICS (Honours) <br> Paper : DSE 1-T 

[CBCS]
Full Marks :60
The figures in the margin indicate full marks.
Candidates are required to give their answers
in their own words as far as practicable.
[Linear Programming]

1. Answer any ten questions:
$2 \times 10=20$
(a) Define convex set.
(b) What is an extreme point in $E^{n}$ ?
(c) What is the dual of an LPP?
(d) Define the saddle point of a matrix game.
(e) Solve the LPP by graphical method :
$\operatorname{Max} Z=5 x_{1}+3 x_{2}$
Subject to $3 x_{1}+5 x_{2} \leq 15$

$$
\begin{gathered}
5 x_{1}+2 x_{2} \leq 10 \\
x_{1}, x_{2} \geq 0
\end{gathered}
$$

(f) When does a set of vectors form a basis of $E^{n}$ ?
(g) Prove that a hyperplane is a convex set.
(h) Is assignment problem a Linear Programming problem? Justify.
(i) Show that the convex hull of two points $x_{\mathrm{i}}$ and $x_{2}$ is the line segment joining these points.
(j) Explain what is meant by a transportation problem.
(k) Is the solution $x_{1}=-6, x_{2}=0, x_{3}=4$ a basic solution of the following equations?

$$
\begin{aligned}
& x_{1}+2 x_{2}+3 x_{3}=6 \\
& 2 x_{1}+x_{2}+4 x_{3}=4
\end{aligned}
$$

(l) State the fundamental theorem of LPP.
(m) When a LPP is said to be has an unbounded solution?
(n) Show that the LPP $\operatorname{Max} z=3 x_{1}+9 x_{2}$

$$
\begin{array}{ll}
\text { subject to } \begin{array}{c}
x_{1}+4 x_{2} \leq 8 \\
\\
x_{1}+2 x_{2} \leq 4 \\
\\
x_{1}, x_{2} \geq 0
\end{array}, ~
\end{array}
$$

admits of a degenerate basic feasible solution.
(o) Write down the transportation problem


## 2. Answer any four questions :

(a) Given a basic feasible solution $X_{B}=B^{-1} b$ with $Z_{0}=C_{B} X_{B}$ to the LPP Max $Z=C X$ subject to $A X=b, X \geq 0$ and $z_{j}-c_{j} \geq 0$ for every column $a_{j}$ in $A$. Prove that $z_{0}$ is the maximum value of $Z$.
(b) Show that by the simplex method, the following LPP admits more than one optimum solution.

$$
\begin{aligned}
\operatorname{Max} Z= & 2 x_{1}+3 x_{2} \\
& x_{1}+3 x_{2} \leq 21 \\
& 2 x_{1}+3 x_{2} \leq 24 \\
& x_{1}+x_{2} \leq 10 \\
& 5 x_{1}+4 x_{2} \leq 48 \\
& x_{1} \geq 0, x_{2} \geq 0
\end{aligned}
$$

(c) Prove that if the primal problem has an unbounded objective function then the dual has no feasible solution.
(d) Using two phase method, show that feasible solution does not exist to the problem

$$
\begin{aligned}
\operatorname{Min} Z= & x_{1}+x_{2} \\
\text { subject to } & 3 x_{1}+2 x_{2} \geq 30 \\
& 2 x_{1}+3 x_{2} \geq 30 \\
& x_{1}+x_{2} \leq 5, x_{1} \geq 0, x_{2} \geq 0 .
\end{aligned}
$$

P.T.O.

## ( 4 )

(e) Formulate the dual of the following LPP and hence solve it.

Maximize $Z=3 x_{1}-2 x_{2}$ subject to $\quad x_{1} \leq 4$
$x_{2} \leq 6$
$x_{1}+x_{2} \leq 5$
$-x_{2} \leq-1$
$x_{1}, x_{2} \geq 0$
(f) Solve the following game graphically
B

| 1 | 3 | 0 | 2 |
| :---: | :---: | :---: | :---: |
| 3 | 0 | 1 | -1 |

3. Answer any two questions:
(a) (i) Use Charne's Big-M method to

Minimize $Z=2 x_{1}+4 x_{2}+x_{3}$

$$
\text { subject to } \begin{array}{r}
x_{1}+2 x_{2}-x_{3} \leq 5 \\
2 x_{1}-x_{2}+2 x_{3}=2 \\
-x_{1}+2 x_{2}+2 x_{3} \geq 1 \\
\\
x_{1}, x_{2}, x_{3} \geq 0
\end{array}
$$

(ii) Determine the position of the point $(-6,1,7,2)$ relative to the hyperplane

$$
\begin{equation*}
3 x_{1}+2 x_{2}+4 x_{3}+6 x_{4}=7 \tag{2}
\end{equation*}
$$

(b) Solve the following transportation problem


Is there any alternative optimal solution to the problem?
$8+2$
(c) (i) Find the assignments to find the minimum cost for the assignment problem with the following cost matrix.

|  | A | B | C | D | E |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | 6 | 5 | 8 | 11 | 16 |
| 2 | 1 | 13 | 16 | 1 | 10 |
| 3 | 16 | 11 | 8 | 8 | 8 |
| 4 | 9 | 14 | 12 | 10 | 16 |
| 5 | 10 | 13 | 11 | 8 | 16 |

(ii) Write a short note on degeneracy in LPP. 3
(d) (i) Prove that if a fixed number $P$ is added to
P.T.O.

## ( 6 )

each element of the pay-off matrix then the value of the game is increased by $P$ while the optimal strategies remains unchanged.
(ii) Use dominance to reduce the pay-off matrix

| A |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: |
| -5 3 1 20 <br> 5 5 4 6 <br> -4 -2 0 -5 |  |  |  |  |

and hence solve.

2022

# 5th Semester Examination 

MATHEMATICS (Honours)

## Paper: DSE 2-T

[CBCS]

## Full Marks : 60

Time : Three Hours
The figures in the margin indicate full marks.
Candidates are required to give their answers in their own words as far as practicable.

## [Probability and Statistics]

## Group - A

1. Answer any ten questions:
$2 \times 10=20$
(a) A freshman class at a college has 200 students of which 150 are women and 50 are majoring in maths, and 25 maths major are women. If a student is selected at random from the freshman class, what is the probability that the student will be either a mathematics major or a women?
(b) $A$ speaks the truth in $75 \%$ cases and $B$ in $80 \%$ cases. In what percentage of cases are they likely to contradict each other in stating the same fact?
(c) Write the pdf of Gamma distribution and its mean and variance.
(d) Find $E(X)$ for the following density function:

$$
f(x)= \begin{cases}\frac{4 x}{5}, & 0<x \leq 1 \\ \frac{2}{5}(3-x), & 1<x \leq 2 \\ 0, & \text { elsewhere }\end{cases}
$$

(e) Accidents take place in a factory at a rate of 6 per year. What is the probability that there is no accident in a given month?
(f) $X, Y, Z$ are three random variables, with $\sigma_{x}=2$, $\sigma_{y}=1$ and $\sigma_{z}=3 ; \rho_{x y}=0.3, \rho_{y z}=0.5$ and $\rho_{x}=0.5$. Find the variance of $U^{0}=X+Y-Z$.
(g) If the lines of regression of $y$ on $x$ and $x$ on $y$ are $3 x+2 y=26$ and $6 x+y=31$, respectively. Find the correlation coefficient between $x$ and $y$.
(h) Let $U$ and $V$ be two random variables with $E(U)=0=E(V), \operatorname{var}(U)=\operatorname{var}(V)=1$. Then prove that $-1 \leq E(U V) \leq 1$. $0 \% / 63$
(i) State weak and strong law of large numbers.
(j) Let $X=\left(X_{1}, X_{2}, \cdots, X_{54}\right)$ be a random sample from a discrete distribution with pmf $p(x)=\frac{1}{3}$,
$x=2,4,6$. Find the probability distribution of sample mean $\bar{X}$ using central limit theorem.
(k) Let $X_{1}, X_{2}, \cdots, X_{n}$ be independent and identically $N\left(\mu, \sigma^{2}\right)$ distributed. Find method of moment estimator of $\mu, \sigma^{2}$.
(l) The bivariate random variable $(X, Y)$ jointly follow the probability density function

$$
f(x, y)=\left\{\begin{array}{l}
k x^{2}(8-y), x<y<2 x, 0 \leq x \leq 2 \\
0, \quad \text { elsewhere }
\end{array}\right.
$$



Find the $k$.
(m) Let $X_{1}, X_{2}, \cdots, X_{n}$ be a random sample from $N\left(\mu, \sigma^{2}\right)$. Find the sampling distribution of $W=\sum_{i=1}^{n}\left(\frac{X_{i}-\mu}{\sigma}\right)^{2}$.
(n) Let $X$ be a random variable follows $N(800,144)$ distribution. Find $P(X<772)$. Given that $P(Z<2.33)=0.0099$, where $Z$ follows standard normal distribution.
(o) Define Markov chain with an example.

## ( 4 ) <br> Group - $\mathbb{B}$

2. Answer any four questions :
(a) Let $X \sim \operatorname{Bin}(n, p)$ and $Y=\frac{X-n p}{\sqrt{n p q}}$. Prove that the distribution of $Y$ converges to $N(0,1)$ as $n \rightarrow \infty$ (not using Central limit theorem).
(b) State and prove Chapman-Kolmogorov equation.
(c) Let $X_{1}, X_{2}, \cdots, X_{n}$ be independent and identically $N\left(\mu, \sigma^{2}\right)$ distributed. Find method of moment estimator of $\mu, \sigma^{2}$ by calculating raw moments.
(d) Find the value of $k$ so that the following table may represent a joint distribution

|  | $Y=1$ | $Y=2$ |
| :---: | :---: | :---: |
| $X=1$ | 0.4 | 0.1 |
| $X=2$ | $k$ | 0.3 |

Find conditional distribution of $X$ given $Y=y$ and also find conditional expectation of $X$ given $Y=y$.
(e) A die is thrown 3600 times, show that the probability that the number of sixes lies between 550 and 650 is at least $4 / 5$ (use Chebyshev's inequality).
(f) Bearings made from a certain process have a mean diameter 0.0566 cm and a standard deviation 0.004 cm . Assuming that the data may be looked upon as a random sample from a normal population, construct a $95 \%$ confidence interval for the actual average diameter of bearings made by the process. Given that $P(l>2.262)=0.025$ with 9 degrees of freedom and $P(t>2.228)=0.025$ with 10 degrees of freedom.

## Group - C

3. Answer any two questions :
$10 \times 2=20$
(a). (i) What is called likelihood function?
(ii) Let $X_{1}, X_{2}, \cdots, X_{n} \sim U(a, b)$. Find maximum likelihood estimators of $a$ and $b$.
(iii) A random sample of size 25 is taken from a Poisson distribution with the parameter $\lambda$. If the sum of all observations is 150 , what is the method of moment estimate of $\lambda$ ? - $2+3+5$
(b) Following are the mileages recorded (km per litre of petrol) in 16 runs of a new model of car: 22.16, 22.37, 22.50, 22.04, 22.25, 23.01, 22.81, 22.63, 23.18, 22.55, 22.75, 22.95, 22.50, 22.38, 23, 22.17.

Assuming the mileage follows a normal distribution with mean $\mu$ and variance $\sigma^{2}$, test the hypotheses
(i) $H: \mu=22.5$ vs. $H_{1}: \mu \neq 22.5$ and
(ii) $H: \sigma^{2} \leq 0.3$ vs. $H_{1}: \sigma^{2}>0.3$.

Take level of significance 0.05 .
Given that $t_{0.025,15}=2.131, t_{0.05,15}=1.753$, $\chi_{0.05,15}^{2}=24.996, \chi_{0.025,15}^{2}=27.488$, choose the appropriate. $5+5$
(c) The joint density function of $(X, Y)$ is given by $f(x, y)=\left\{\begin{array}{ll}10 x y^{2}, & 0<x<y<1 \\ 0, & \text { elsewhere }\end{array}\right.$. Find the marginal and conditional probability density functions of $X$ and $Y$. Also find $\mu_{X}, \mu_{Y}, \sigma_{X}^{2}, \sigma_{Y}^{2}, \operatorname{cov}(X, Y)$ and $\rho(X, Y)$. $E(x-1)_{m x}=$
(d) (i) Let $F(x)$ be the distribution function of a continuous random variable $X$. Show that the expectation of $X$ can be expressed as

$$
E(X)=\int_{x=0}^{\infty}\{1-F(x)-F(-x)\} d x .
$$

(ii) For any random variable $X$ (discrete or continuous) and for any real number $c$, prove
that $E(|X-c|) \geq E(|X-\mu|)$ provided the expectations exist and $\mu$ is the median of $X$.
(iii) If $X$ is $\gamma(l)$ variate, then compute $E(\sqrt{X})$. $4+4+2$

